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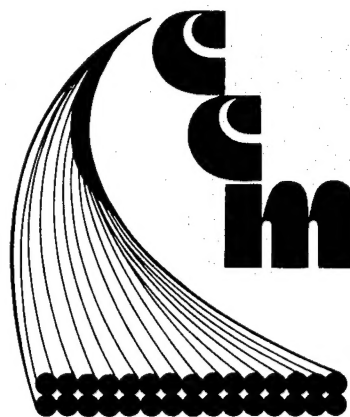
AXISYMMETRIC ANALYSIS OF THE CYLINDRICALLY
ORTHOTROPIC DISK OF
VARIABLE FIBER ORIENTATION

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BY

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Axisymmetric Analysis of the Cylindrically
Orthotropic Disk of Variable Fiber Orientation

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Prepared for

Rogers Corporation

Lurie Research and Development Center

Rogers, Connecticut

April 1, 1979

Axisymmetric Analysis of the Cylindrically Orthotropic Disk of Variable Fiber Orientation

Abstract

Injection molding of axisymmetric bodies with fiber-reinforced molding compounds results in cylindrically orthotropic components in which the fiber orientation varies with radial position. Consequently, development of analysis techniques for determining the effect of material property variability on the response of these components is of great importance. In the present study, a numerical integration scheme for the analysis of cylindrically orthotropic annular disks with variable elastic constants is presented. The influence of material property variations, temperature changes and centrifugal forces on the response of an annular disk subjected to internal and external pressure or displacement boundary condition is included in the analysis. Correlation of numerical integration results with analytical and finite-element solutions is excellent.

Axisymmetric Analysis of the Cylindrically Orthotropic Disk of Variable Fiber Orientation

Introduction

Injection molding of axisymmetric bodies with fiber-reinforced molding compounds results in cylindrically orthotropic components. In general, the fiber orientation will not be constant throughout the body, but will be a function of radial position. It has recently been shown [1] that the flow field and mold geometry determine fiber orientation. For example, the influence of converging and diverging flow fields on fiber orientation are shown clearly in Figure 1. Consequently, injection molding of an axisymmetric mold uniformly on the inner or outer radius (axisymmetric flow) will result in fiber orientation which is dependent solely on the radial position. Material properties are determined uniquely by fiber orientation, constituent properties, fiber aspect ratio and fiber volume fraction. McCullough [1, 2] has introduced the following parameters to quantify fiber orientation:

$$f = 1/2 [3 \langle \cos^2 \phi \rangle - 1]$$

$$g = 1/4 [5 \langle \cos^4 \phi \rangle - 1]$$

$$\langle \cos^m \phi \rangle = \int_0^{\pi/2} N(\phi) \cos^m \phi \sin \phi d\phi$$

where ϕ is the angle which a fiber makes with the longitudinal

direction and $N(\phi)$ is the percentage of fibers with that direction. In figure 2, the bounds on Young's modulus as a function of the orientation parameter "f" are shown. Clearly, the variation of fiber orientation with radial position significantly influences the mechanical properties of the component.

In the present study, a numerical integration scheme is developed to determine the response of cylindrically orthotropic disks with elastic constants which vary in the radial direction. Uniform temperature variations and centrifugal body forces, as well as, pressure on displacement boundary conditions prescribed on the inner and outer radii are considered.

Formulation of the governing equation in terms of displacements yields a second order linear ordinary differential equation with non-constant coefficients (See Appendix A). In general, a closed form solution of this equation does not exist. The integration scheme requires the reduction of the governing equation to two simultaneous first order differential equations which are solved using Hamming's Predictor-Corrector Method [3]. Unfortunately, Hamming's Method requires two initial value conditions (one for each first order equation) whereas only one is known in the actual boundary value problem. Consequently, a half-interval search technique is incorporated into the program in which upper and lower bounds for the unknown initial condition are

prescribed. The average value is employed in the integration and correlation of the solution with the second known boundary condition enables the interval of uncertainty to be halved. Subsequent iterations converge quickly to the solution. In fact, if Δ_1 is the length of the starting interval, then the number (N) of interval halving operations required to reduce the interval of uncertainty to Δ_N is given by

$$N = \frac{\ln(\Delta_1/\Delta_N)}{\ln 2}$$

Closed-form solutions for two special variations of elastic properties are presented in Appendices B and C to verify numerical integration results. The first solution is for uniform properties and the second assumes that the moduli vary along a radius according to a power law:

$$E_r = E_{rm} r^m, E_\theta = E_{\theta m} r^m$$

where m is an arbitrary real number, and the Poisson's ratios are held constant. In addition, finite element results for a linear variation of properties are compared to the numerical integration results.

Correlation of Results

Agreement of numerical integration results with analytical and finite element results was found to be excellent. In Figures 3-6, typical results for temperature variations, centrifugal body forces and internal and external radial stress tractions are presented. The solid line in all figures corresponds to the numerical integration prediction. Superimposed are analytic and finite element results shown as symbols. The excellent agreement is obtained using an integration step-size of 0.001 inches (0.025 mm). Approximately 15-20 iterations are required to determine the unknown initial condition with sufficient accuracy to satisfy the remaining boundary condition on the outer radius. Although only the correlation of stress components are presented in Figures 3-6, excellent agreement was obtained for displacement and strain component values as well.

For reference, the material properties employed in the various solutions are given below:

Constant Properties: Radial Fiber Orientation

$$\begin{aligned} E_r &= 2.6 \text{ Msi} (17.9 \text{ GPa}) & \alpha_r &= 4.5 \times 10^{-6} / ^\circ\text{F} \quad (8.1 \times 10^{-6} / ^\circ\text{C}) \\ E_\theta &= 1.4 \text{ Msi} (9.65 \text{ GPa}) & \alpha_\theta &= 10.1 \times 10^{-6} / ^\circ\text{F} \quad (18.2 \times 10^{-6} / ^\circ\text{C}) \\ \nu_{\theta r} &= 0.27 \end{aligned}$$

Constant Properties: Tangential Fiber Orientation

$$\begin{aligned} E_r &= 1.4 \text{ Msi} (9.65 \text{ GPa}) & \alpha_r &= 10.1 \times 10^{-6} / ^\circ\text{F} \quad (18.2 \times 10^{-6} / ^\circ\text{C}) \\ E_\theta &= 2.6 \text{ Msi} (17.9 \text{ GPa}) & \alpha_\theta &= 4.5 \times 10^{-6} / ^\circ\text{F} \quad (8.1 \times 10^{-6} / ^\circ\text{C}) \\ \nu_{\theta r} &= .501 \end{aligned}$$

Power Law Variation (m specified)

$$E_r = 2.6 r^m \text{ Msi } (17.9 r^m \text{ GPa}) \quad \nu_{r\theta} = .501$$

$$E_\theta = 1.4 r^m \text{ Msi } (9.65 r^m \text{ GPa}) \quad \alpha_r = (4.5 \times 10^{-6}) r^m / ^\circ\text{F} \\ (8.1 \times 10^{-6}) r^m / ^\circ\text{C}$$

$$\nu_{\theta r} = .27 \quad \alpha_\theta = (10.1 \times 10^{-6}) r^m / ^\circ\text{F} \\ (18.2 \times 10^{-6}) r^m / ^\circ\text{C}$$

Linear Variation

$$E_r = \left(\frac{-1.2}{(b-a)} (r-a) + 2.6 \right) \text{ Msi } \left(\left(\frac{-8.27}{(b-a)} (r-a) + 17.9 \right) \text{ GPa} \right)$$

$$E_\theta = \left(\frac{1.2}{(b-a)} (r-a) + 1.4 \right) \text{ Msi } \left(\left(\frac{8.27}{(b-a)} (r-a) + 9.65 \right) \text{ GPa} \right)$$

$$\nu_{\theta r} = .27$$

$$\nu_{r\theta} = \nu_{\theta r} E_r / E_\theta$$

$$\alpha_r = \left(\frac{5.6(r-a)}{(b-a)} + 4.5 \right) \times 10^{-6} / ^\circ\text{F} \left(\left(\frac{10.1(4-a)}{(b-a)} + 8.1 \right) \times 10^{-6} / ^\circ\text{C} \right)$$

$$\alpha_\theta = \left(\frac{-5.6(r-a)}{(b-a)} + 10.1 \right) \times 10^{-6} / ^\circ\text{F} \left(\left(\frac{-10.1(r-a)}{(b-a)} + 18.2 \right) \times 10^{-6} / ^\circ\text{C} \right)$$

where

E_r - Young's Modulus in radial direction

E_θ - Young's Modulus in tangential direction

$\nu_{r\theta}, \nu_{\theta r}$ - Poisson ratios

α_r - Thermal coefficient of expansion in radial direction

α_θ - Thermal coefficient of expansion in tangential direction

User's Guide

Fortran computer codes have been developed for two analytical solutions and for the Hamming's Predictor-Corrector Method. Analysis details and program listings may be found in Appendices A, B and C. The programs have been written in an interactive format which necessitates execution from a terminal or similar device. In Table 1, program symbol definitions are defined. The following examples will be illustrative.

Analytic Solution: Constant Properties (VARPROP/CFD)

Table 2 indicates the line numbers in VARPROP/CFD which describe the material properties. These are the only lines that must be altered when another set of properties are to be input. In Figure 7 a sample program execution is presented where data input is requested by the program. Note that displacement boundary conditions are not possible.

Analytic Solution: Power Law Variation (VARPROP/CF2)

The analytic solution assumes the following material property variation:

$$\begin{array}{ll} E_r = E_{rm} r^m & \nu_{r\theta} = \nu_{\theta r} E_{rm}/E_{\theta m} \\ E_{\theta} = E_{\theta m} r^m & \alpha_r = \alpha_{rm} r^m \\ \nu_{\theta r} = \text{constant} & \alpha_{\theta} = \alpha_{\theta m} r^m \end{array}$$

In Table 3, the line numbers in VARPROP/CF2 which describe the material properties are shown. Execution is straightforward as indicated in Figure 8. Note that centrifugal body forces and displacement boundary conditions are not included.

Numerical Integration (VARPROP/NUMD)

The material property section of VARPROP/WUMD is located in Subroutine PROP (See Appendix A for further detail). All property dependence on radial position must be input. Derivatives of these properties are also required. In Table 4, for example, the material properties employed in the correlation of numerical with analytical results (constant properties) are presented. Material property input for power law and linear variations are shown in Tables 5 and 6, respectively. Note that derivatives are input in a straightforward manner.

Execution of VARPROP/NUMD is also in an interactive mode. The program versatility allows displacement or stress boundary conditions in the inner and outer surface. As mentioned previously, upper and lower bounds on the unknown initial condition at the inner radius must be supplied. If a radial stress initial condition is specified, bounds on the tangential stress components are input as shown in Figures 9 and 10. For displacement initial conditions, bounds on the radial strain component at the inner radius

are required (See Figures 11 and 12). Results from numerical integration and the analytical solution for constant properties (radial fiber orientation) are presented in Figures 7 and 9. Comparison of solutions illustrates the excellent accuracy obtained with the Hamming's Predictor -Corrector Method.

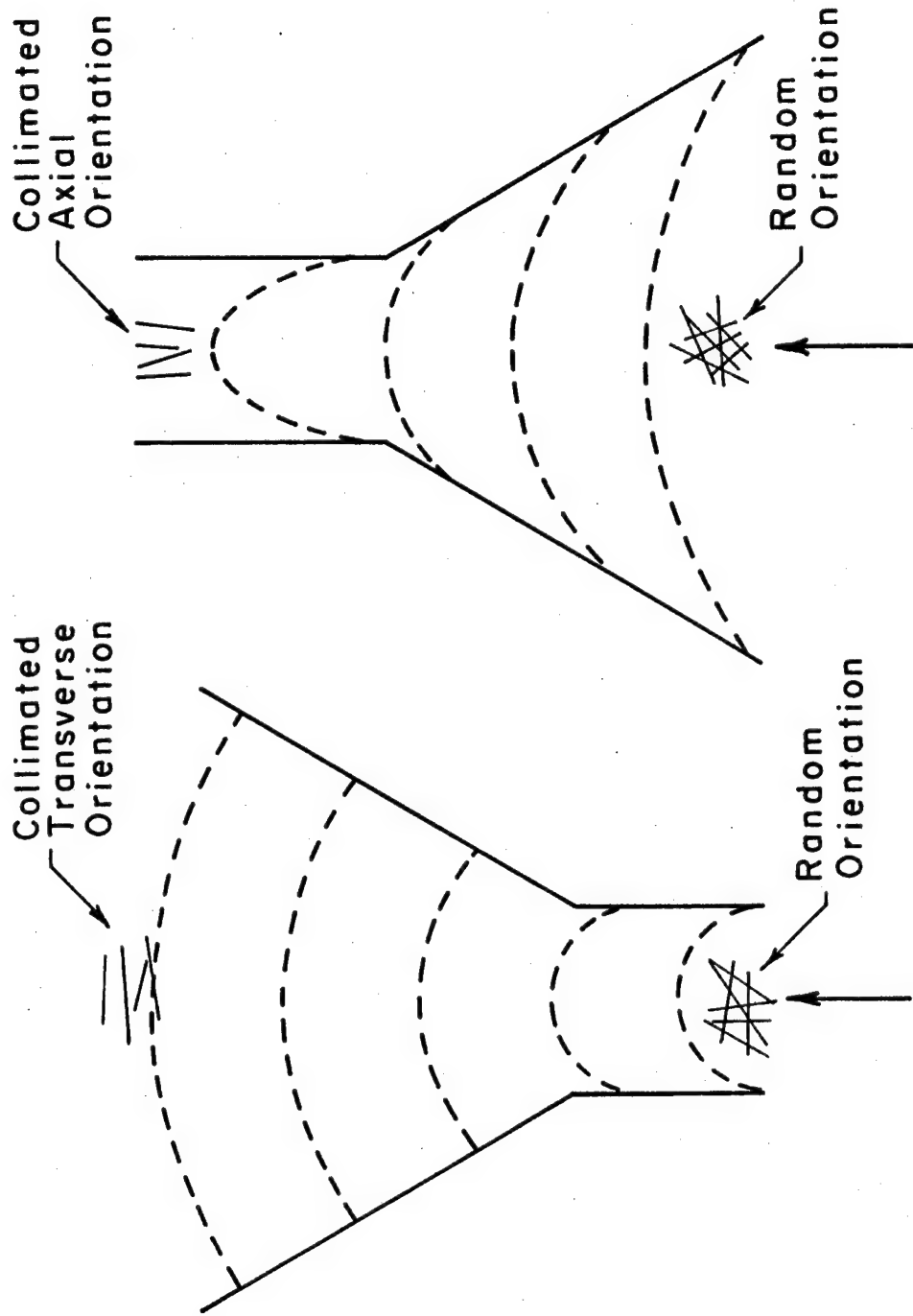


Figure 1. Fiber orientations of converging and diverging flow fields

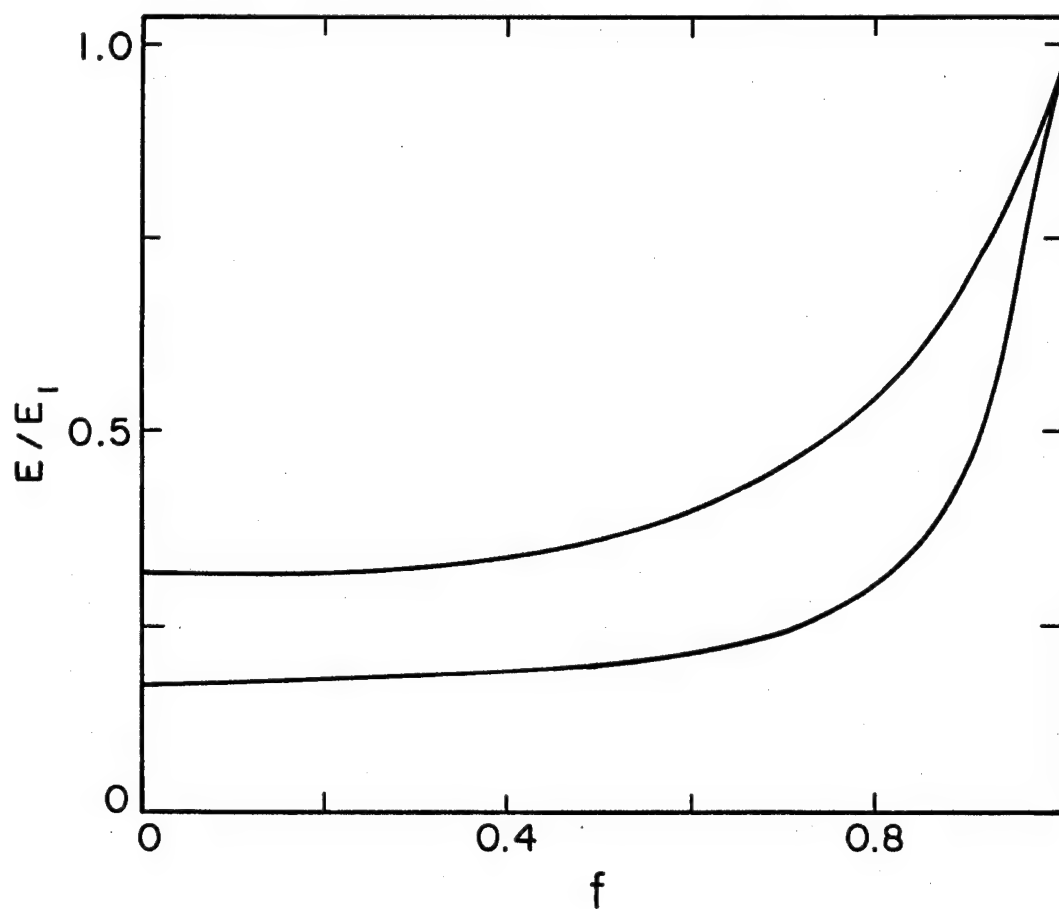


Figure 2 Dependence of Young's modulus on fiber orientation.

Figure 3. Correlations of Numerical Integration With Analytical and Finite Element Results for Temperature Variations

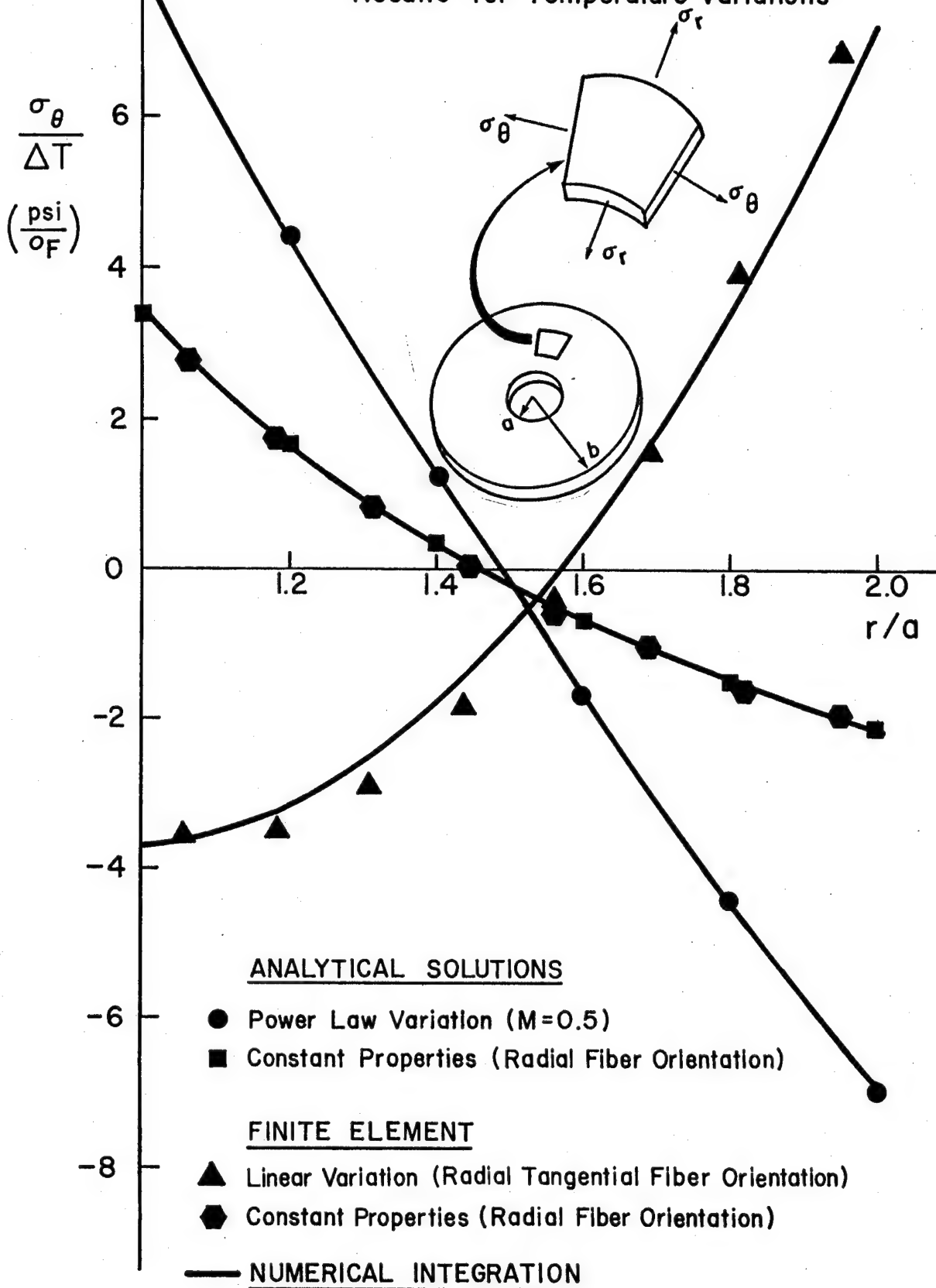


FIGURE 4. Correlation of Numerical Integration Results
With Analytical Solution For Centrifugal
Body Forces

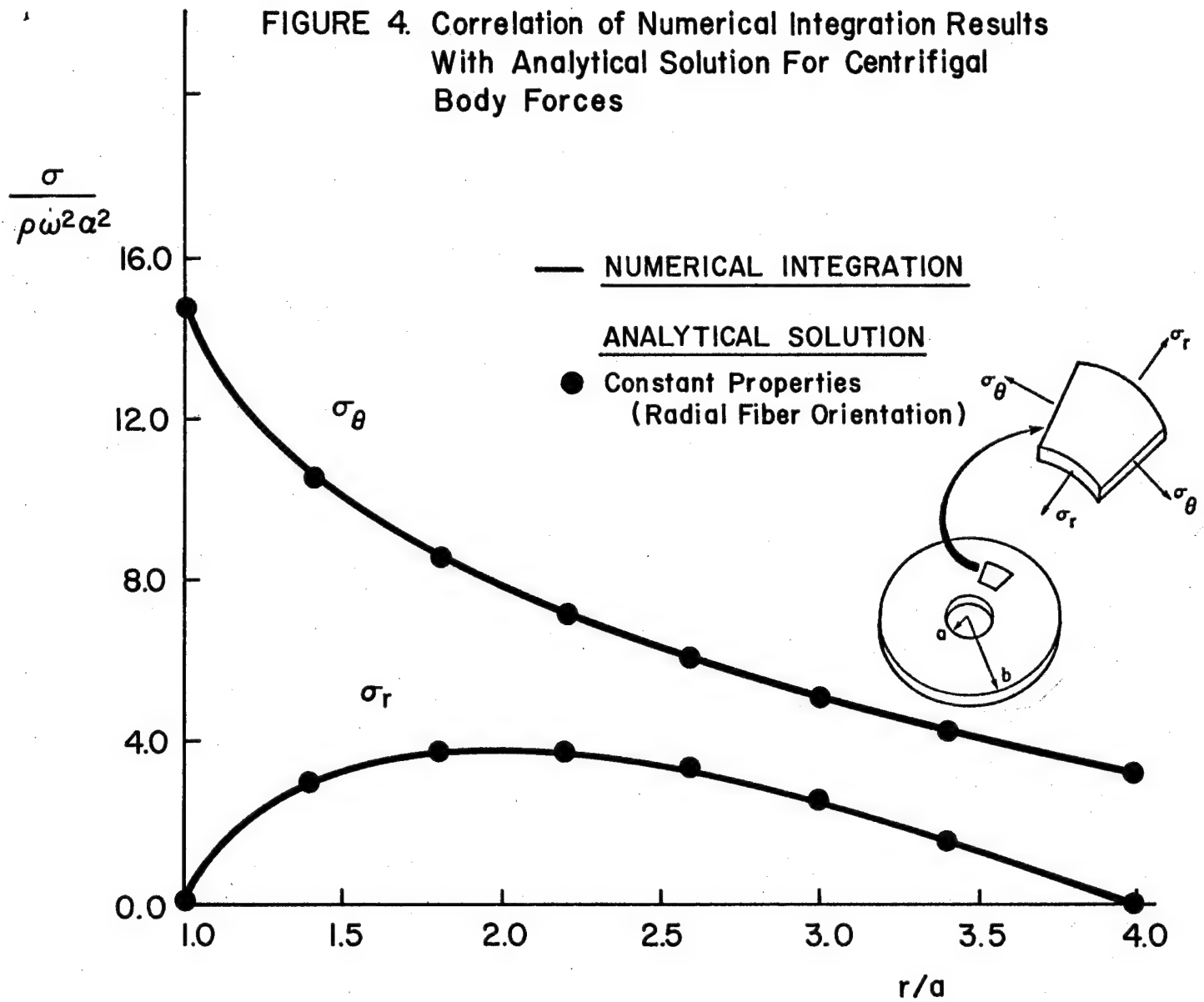


FIGURE 5. Correlation of Numerical Integration Results With Analytical Solution For Radial Traction On Inner Surface

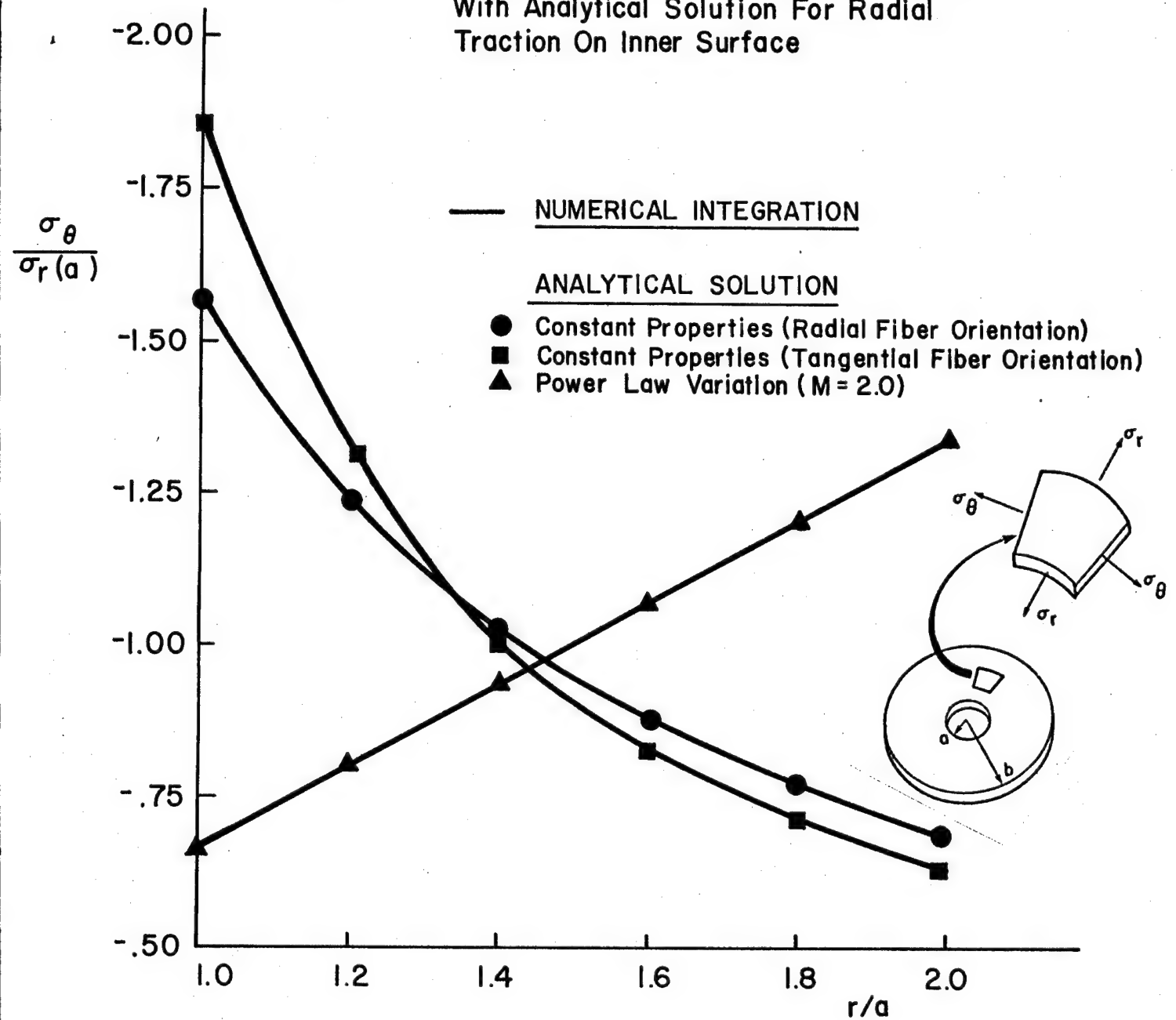


Figure 6. Correlation of Numerical Integration Results With Analytical Solution For Radial Stress Traction On Outer Surface

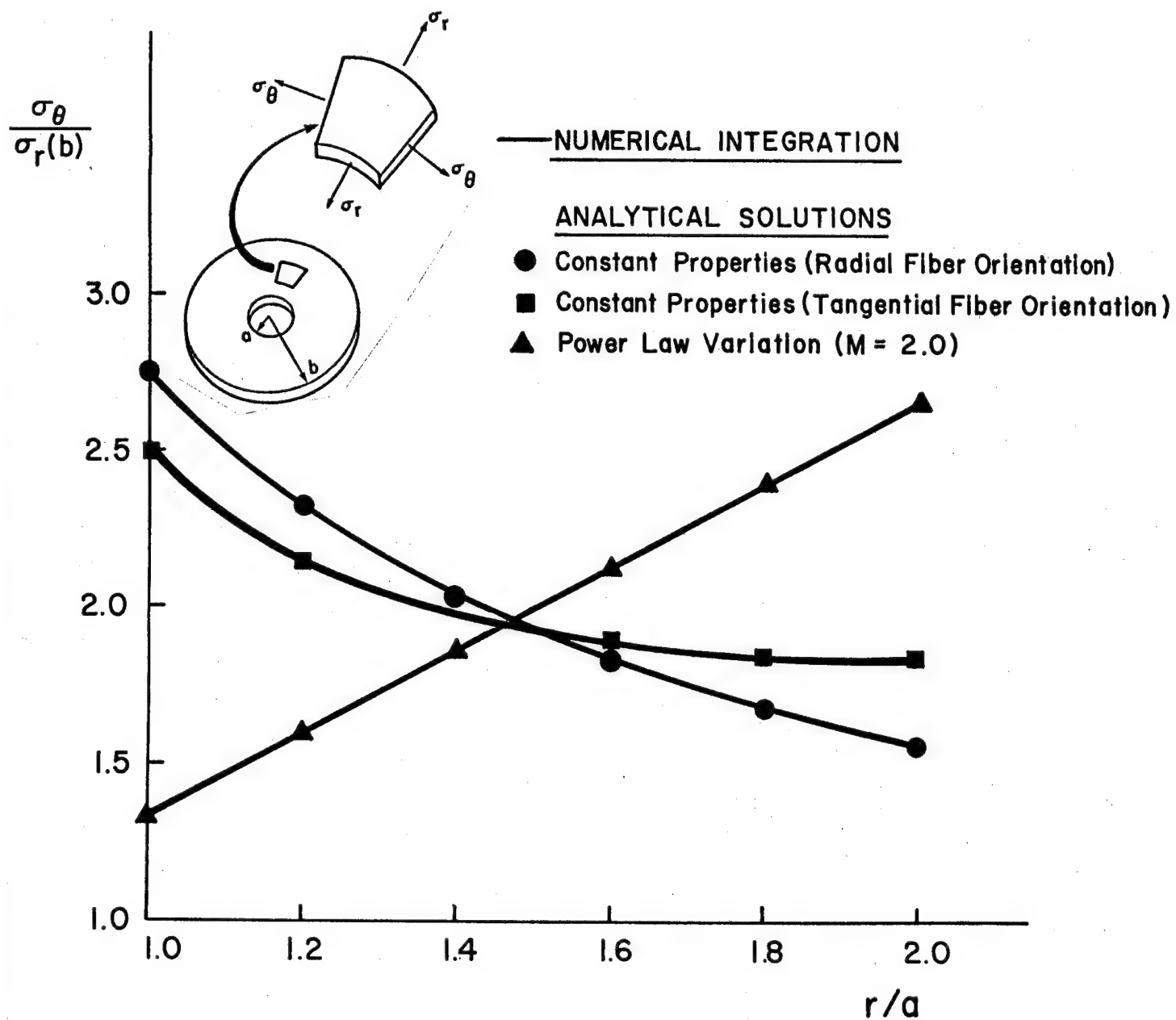


Table 1 Symbol Definitions

LIST OF SYMBOLS:

XI	INITIAL STARTING VALUE FOR INDEPENDENT VARIABLE
H	INTEGRATION STEP-SIZE
XMAX	UPPER LIMIT OF INTEGRATION
N	NUMBER OF INITIAL CONDITIONS
YR(I)	N INITIAL CONDITIONS AT X
FR(I)	DERIVATIVES IN ORIGINAL SYSTEM OF DIFFERENTIAL EQUATIONS
Y(I,J)	Y VALUE AT ITH X VALUE FOR JTH DIFFERENTIAL EQUATION
F(I,J)	DERIVATIVE AT ITH X VALUE FOR JTH DIFFERENTIAL EQUATION
TE(I)	TRUNCATION ERROR FOR ITH CORRECTOR EQUATION
INT	NUMBER OF INTEGRATIONS BETWEEN OUTPUT
NN	UPPER LIMIT ON THE NUMBER OF HALF-INTERVAL ITERATIONS
Y2LEFT	LOWER LIMIT ON UNKNOWN INITIAL VALUE
Y2RITE	UPPER LIMIT ON UNKNOWN INITIAL VALUE
NHIS	NUMBER OF HALF-INTERVAL ITERATIONS BETWEEN PRINTOUT
SGI	RADIAL STRESS B.C. ON INNER RADIUS
SGO	RADIAL STRESS B.C. ON OUTER RADIUS
SIGNL	HAS THE VALUE -1, IF $Y1(XMAX) < SGO$ WHEN $Y2(XI) = Y2LEFT$; OTHERWISE THE VALUE IS 1
DT	TEMPERATURE DIFFERENCE RELATIVE TO STRESS FREE STATE
W	ROTATIONAL SPEED(RPM)
RHO	DENSITY(#/IN**3)

Table 2 Material Property Section of VARPROP/CFD

```

3400 C    INPUT MATERIAL PROPERTIES
3500 C    (T: TANGENTIAL, R:RADIAL):
3600 C
3700      ET=1.4E6
3800      VTR=.27
3900      ER=2.6E6
4000      VRT=VTR*ER/ET
4100      AT=10.1E-6
4200      AR=4.5E-6
4300      RHO=.1
4400 C
4500 C    END MATERIAL PROPERTY DESCRIPTION
4600 C
#

```

Table 3 Material Property Section of VARPROP/CF 2

```

3100 C
3200 C INPUT MATERIAL PROPERTIES
3300 C (T: TANGENTIAL, R:RADIAL):
3400 C
3500     ETM=1.4E6
3600     VTR=.27
3700     ERM=2.6E6
3800     ATM=10.1E-6
3900     ARM=4.5E-6
4000 C
4100 C END MATERIAL PROPERTY DESCRIPTION
4200 C
#

```

Table 4 Material Property Section of VARPROP/NUMD
(Constant Properties)

```

31100 C
31200 C INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION
31300 C
31400     ET=1.4E6
31500     VTR=.27
31600     ER=2.6E6
31700     VTRP=0.
31800     AT=10.1E-6
31900     AR=4.5E-6
32000     ERP=0.
32100     ETP=0.
32200     ATP=0.
32300     ARP=0.
32400     VRT=ER*VTR/ET
32500     VRTP=0.
32600 C END MATERIAL PROPERTY INPUT
#

```

Table 5 Material Property Section of VARPROP/NUMD
(Power Law Variation)

```

27504 C
27505 C INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION
27506 C
27600     ETM=1.4E6
27800     VTR=.27
27900     VTRP=0.
28000     ERM=2.6E6
28006     VRT=ERM*VTR/ETM
28008     V RTP=0.
28100     ATM=10.1E-6
28300     ARM=4.5E-6
28310     ET=ETM**X**2
28312     ER=ERM**X**2
28314     AR=ARM**X**2
28316     AT=ATM**X**2
28318     ERP=2.*ERM*X
28319     ETP=2.*ETM*X
28320     ATP=2.*ATM*X
28321     ARP=2.*ARM*X
28322 C END MATERIAL PROPERTY INPUT
#

```

Table 6 Material Property Section of VARPROP/NUMD
(Linear Variation)

```

31100 C
31200 C INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION
31300 C
31400     ETM=1.4E6
31500     VTR=.27
31700     ERM=2.6E6
31900     VTRP=0.
32000     ATM=10.1E-6
32100     ARM=4.5E-6
32200     ET=(ERM-ETM)*X/(XMAX-XI)+ETM-(ERM-ETM)*XI/(XMAX-XI)
32300     ER=(ERM-ERM)*X/(XMAX-XI)+ERM-(ETM-ERM)*XI/(XMAX-XI)
32400     AR=(ATM-ARM)*X/(XMAX-XI)+ARM-(ATM-ARM)*XI/(XMAX-XI)
32500     AT=(ARM-ATM)*X/(XMAX-XI)+ATM-(ARM-ATM)*XI/(XMAX-XI)
32600     ERP=(ETM-ERM)/(XMAX-XI)
32700     ETP=(ERM-ETM)/(XMAX-XI)
32800     ATP=(ARM-ATM)/(XMAX-XI)
32900     ARP=(ATM-ARM)/(XMAX-XI)
32950     VRT=ER*VTR/ET
32960     VRTP=VTR*ERP/ET-VTR*ER/ET**2*ETP
33000 C END MATERIAL PROPERTY INPUT
#

```

Figure 7 Typical Output of VARPROP/CFD

```

RUN VARPROP/CFD
#RUNNING 5396
ENTER XI,XMAX,DX,DT,W
1,2,.2,100,10000
#?
ENTER SGI,SGO
100,50

XI      =      .100E+01
XMAX    =      .2000E+01
SGO     =      .5000E+02
SGI     =      .1000E+03
ET      =      .1400E+07
ER      =      .2600E+07
AT      =      .1010E-04
AR      =      .4500E-05
VTR     =      .270E+00
DT      =      .1000E+03
W       =      .1000E+05
RHO     =      .1000E+00

X
1.0000
1.2000
1.4000
1.6000
1.8000
2.0000

#ET=56.2 FT=0.2 IO=0.2

      SIGR      SIGT      EPR      EPT      U
      .1000000E+03      .1316823E+04      .2345029E-03      .1931302E-02      .1931302E-02
      .2143555E+03      .9662531E+03      .3460956E-03      .1658841E-02      .1990609E-02
      .2343291E+03      .7153699E+03      .4021624E-03      .1475786E-02      .2066101E-02
      .2019383E+03      .5212714E+03      .4271377E-03      .1343391E-02      .2149426E-02
      .1370740E+03      .3618789E+03      .4329298E-03      .1242049E-02      .2235689E-02
      .5000000E+02      .2247240E+03      .4258911E-03      .1160874E-02      .2321749E-02

```

Figure 8 Typical Output of VARPROP/CF 2

```

R VARPROP/CF2
#RUNNING 5339
ENTER XI,XMAX,DX,DT
#?
1,2,.2,100
ENTER SGI,SGO,M
100,50,2

XI      = .100E+01
XMAX    = .2000E+01
SGO     = .5000E+02
SGI     = .1000E+03
ETM     = .1400E+07
ERM     = .2600E+07
ATM     = .1010E-04
ARM     = .4500E-05
VTR     = .270E+00
DT      = .1000E+03
M       = .20000000E+01

X      SIGR      SIGT      EPR      EPT      U
1.0000 .10000000E+03 .3772882E+04 -.2391657E-03 .3685630E-02 .3685630E-02
1.2000 .6970974E+03 .3475520E+04 .3687192E-03 .3085007E-02 .3702008E-02
1.4000 .1024690E+04 .2341979E+04 .8526347E-03 .2732265E-02 .3825171E-02
1.6000 .1061023E+04 .7181939E+04 .1305998E-02 .2525707E-02 .4041131E-02
1.8000 .7580607E+03 -.3681644E+04 .1767134E-02 .2415627E-02 .4348129E-02
2.0000 .5000000E+02 -.9311337E+04 .2253747E-02 .2374851E-02 .4749701E-02

#ET=1:09.7 PT=0.2 IO=0.2

```

```

RUN VARPROP/NUMD
#RUNNING 5451
ENTER XI,H,XMAX,INT,DT,W(RPM),RHO(LB/IN**3)
#?
1,.001,2,200,100,10000,.1
ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)
100,1
ENTER Y2LEFT,Y2RITE
1000,1500
ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)
50,1
ENTER NN,NHIS,SIGNL
15,15,-1

```

```

H = .100E-02
XI = 1.0000
XMAX = 2.0000
N = 2
INT = 200
YR1 = 100.0000
TIC1 = 1.0000
Y2LEFT = 1000.0000
Y2RITE = 1500.0000
NN = 15
NHIS = 15
SIGNL = -1
DT = 100.0000
RHO = 0.100v
W = 10000.0000

```

Constant Properties (Radial Fiber Orientation)

	X	U	Y	EPR	EPT	SIGR	SIGT
ITERATION	1	LOWER=	1000.0000	UPPER=	1500.0000	UNKNOWN I.C.=	1250.0000
ITERATION	2	LOWER=	1250.0000	UPPER=	1500.0000	UNKNOWN I.C.=	1375.0000
ITERATION	3	LOWER=	1250.0000	UPPER=	1375.0000	UNKNOWN I.C.=	1312.5000
ITERATION	4	LOWER=	1312.5000	UPPER=	1375.0000	UNKNOWN I.C.=	1343.7500
ITERATION	5	LOWER=	1312.5000	UPPER=	1343.7500	UNKNOWN I.C.=	1328.1250
ITERATION	6	LOWER=	1312.5000	UPPER=	1328.1250	UNKNOWN I.C.=	1320.3125
ITERATION	7	LOWER=	1312.5000	UPPER=	1320.3125	UNKNOWN I.C.=	1316.4063
ITERATION	8	LOWER=	1316.4063	UPPER=	1320.3125	UNKNOWN I.C.=	1318.3594
ITERATION	9	LOWER=	1316.4063	UPPER=	1318.3594	UNKNOWN I.C.=	1317.3828
ITERATION	10	LOWER=	1316.4063	UPPER=	1317.3828	UNKNOWN I.C.=	1316.8945
ITERATION	11	LOWER=	1316.4063	UPPER=	1316.8945	UNKNOWN I.C.=	1316.6504
ITERATION	12	LOWER=	1316.6504	UPPER=	1316.8945	UNKNOWN I.C.=	1316.7725
ITERATION	13	LOWER=	1316.7725	UPPER=	1316.8945	UNKNOWN I.C.=	1316.8335
ITERATION	14	LOWER=	1316.7725	UPPER=	1316.8335	UNKNOWN I.C.=	1316.8030
ITERATION	15	LOWER=	1316.8030	UPPER=	1316.8335	UNKNOWN I.C.=	1316.8182
1.20000		.1990606E-02		.3460961E-03	.1658838E-02	.2143548E+03	.9662493E+03
1.40000		.2066098E-02		.4021626E-03	.1475784E-02	.2343280E+03	.7153666E+03
1.60000		.2149423E-02		.4271377E-03	.1343390E-02	.2019369E+03	.5212684E+03
1.80000		.2235686E-02		.4329298E-03	.1242048E-02	.1370724E+03	.3618762E+03
2.00000		.2321746E-02		.4258910E-03	.1160873E-02	.4999837E+02	.2247314E+03

#ET=2:26.3 PT=21.3 IO=0.2

Figure 9 Typical Output of VARPROP/NUMD:
Stress Initial Condition - Stress
Outer Boundary Condition


```

RUN VARPROP/NUMD
BRUNING 5609
ENTER XI,H,XMAX,INI,DT,B(RPM),RHO(LB/IN**3)
#
1,001,2,200,100,10000,1
ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)
100,1
ENTER YZLEFT,YZRITE
0,10000
ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)
.005,2
ENTER NN,NHIS,SIGNL
15,15,-1

```

```

H = .100E-02
XI = 1.0000
XMAX = 2.0000
N = 2
INT = 200
YR1 = 100.0000
YR1 = 1.0000
YZLEFT = 0.0000
YZRITE = 10000.0000
NN = 15
NHIS = 15
SIGNL = -1
DT = 100.0000
RHO = 0.1000
W = 10000.0000

```

X

U

Y

EPR

EPT

SIGR

SIGT

ITERATION	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
LOWER=	0.0000	5000.0000	5000.0000	5000.0000	5000.0000	5000.0000	5312.5000	5312.5000	5312.5000	5312.5000	5312.5000	5317.3828	5317.3828	5318.6035	5318.6035
UPPER=	10000.0000	10000.0000	7500.0000	6250.0000	5625.0000	5625.0000	5468.7500	5390.6250	5351.5625	5332.0313	5322.2656	5322.2656	5319.8242	5319.8242	5319.2139
UNKNOWN I.C.=	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN	UNKNOWN
I.C.=	5000.0000	7500.0000	6250.0000	5625.0000	5312.5000	5468.7500	5390.6250	5351.5625	5332.0313	5322.2656	5317.3828	5319.8242	5318.6035	5319.2139	5318.9087
1.20000															
1.40000															
1.60000															
1.80000															
2.00000															
PT=3:11.1															
IO=0.2															

Figure 10 Typical Output of VARPROP/NUMD:
Stress Initial Condition - Displacement
Outer Boundary Conditions

Constant Properties (Radial Fiber Orientation)

.4331217E+04
.3561579E+04
.3172818E+04
.2795299E+04
.2490236E+04

```

RUN VARPROP/NUMD
#RUNNING 5500
ENTER XI,H,XMAX,INT,DT,W(RPM),RHO(LB/IN**3)
#?
1,.001,2,200,100,10000,.1
ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)
.01,2
ENTER Y2LEFT,Y2RITE
-1,.1
ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)
.05,2
ENTER NN,NHIS,SIGNL
15,15,-1

```

```

H = .100E-02
XI = 1.0000
XMAX = 2.0000
N = 2
INT = 200
YRI = 0.0100
TICI = 2.0000
Y2LEFT = -0.1000
Y2RITE = 0.1000
NN = 15
NHIS = 15
SIGNL = -1
DT = 100.0000
RHO = 0.1000
W = 10000.0000

```

Figure 11 Typical Output for VARPROP/NUMD:
Displacement Initial Condition -
Displacement Outer Boundary Condition

Constant Properties (Radial Fiber Orientation)

ITERATION	X	U	Y	EPR	EPT	SIGR	SIGT
1	LOWER =	-0.1000		UPPER =	0.1000	UNKNOWN I.C. =	0.0000
2	LOWER =	0.0000		UPPER =	0.1000	UNKNOWN I.C. =	0.0500
3	LOWER =	0.0500		UPPER =	0.1000	UNKNOWN I.C. =	0.0750
4	LOWER =	0.0500		UPPER =	0.0750	UNKNOWN I.C. =	0.0625
5	LOWER =	0.0500		UPPER =	0.0625	UNKNOWN I.C. =	0.0563
6	LOWER =	0.0500		UPPER =	0.0563	UNKNOWN I.C. =	0.0531
7	LOWER =	0.0531		UPPER =	0.0563	UNKNOWN I.C. =	0.0547
8	LOWER =	0.0531		UPPER =	0.0547	UNKNOWN I.C. =	0.0539
9	LOWER =	0.0531		UPPER =	0.0535	UNKNOWN I.C. =	0.0535
10	LOWER =	0.0533		UPPER =	0.0535	UNKNOWN I.C. =	0.0533
11	LOWER =	0.0533		UPPER =	0.0535	UNKNOWN I.C. =	0.0534
12	LOWER =	0.0534		UPPER =	0.0535	UNKNOWN I.C. =	0.0535
13	LOWER =	0.0535		UPPER =	0.0535	UNKNOWN I.C. =	0.0535
14	LOWER =	0.0535		UPPER =	0.0535	UNKNOWN I.C. =	0.0535
15	LOWER =	0.0535		UPPER =	0.0535	UNKNOWN I.C. =	0.0535
1.20000		.1987331E-01		.4580194E-01	.1656151E-01	.1490054E+06	.6200357E+05
1.40000		.2849024E-01		.4067638E-01	.2035017E-01	.1366683E+06	.6397668E+05
1.60000		.3624159E-01		.3702574E-01	.2265099E-01	.1275585E+06	.6473819E+05
1.80000		.4336097E-01		.3428931E-01	.2408943E-01	.1204976E+06	.6484556E+05
2.00000		.499722E-01		.3215548E-01	.2499861E-01	.1148191E+06	.6458522E+05

#ET=3:10.2 PT=21.1 IO=0.2

```

RHO VARPROP/NUMD
$ENDING 5765
ENTER XI,H,XMAX,INT,UT,W(RPM),RHO(LB/IN**3)
#?
1E-001,2,200,100,10000,.1
ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)
.005,2
ENTER Y2LEFT,Y2RITE
-.01,.01
ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)
100,1
ENTER NN,NHIS,SIGNL
15,15,-1

```

```

H = .100E-02
XI = 1.0000
XMAX = 2.0000
N = 2
INT = 200
YR1 = 0.0050
YR1C1 = 2.0000
Y2LEFT = -0.0100
Y2RITE = 0.0100
NN = 15
NHIS = 15
SIGNL = -1
DI = 100.0000
RHO = 0.1000
W = 10000.0000

```

X

Y

U

EPR

EPT

SIGR

SIGT

ITERATION	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ITERATION	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER	LOWER
UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER
U	-0.0100	-0.0100	-0.0050	-0.0025	-0.0025	-0.0019	-0.0016	-0.0014	-0.0014	-0.0014	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013
EPR	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER	UPPER
EPT	0.0100	0.0000	0.0000	0.0000	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013
SIGR	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.	UNKNOWN I.C.
SIGT	0.0000	-0.0050	-0.0025	-0.0013	-0.0019	-0.0016	-0.0014	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013	-0.0013
1.20000	.4797600E-02	.4691240E-02	.4644290E-02	.4635376E-02	.4651994E-02										
1.40000															
1.60000															
1.80000															
2.00000															
4ET-2:22.8 PT-21.0 IO-0.2															

Figure 12 Typical Output for VARPROP/NUMD:
Displacement Initial Condition -
Stress Outer Boundary Condition

Conclusions

Development of a numerical integration scheme for the analysis of cylindrically orthotropic annular disks with variable elastic constants has been accomplished. The integration scheme utilizes Hamming's Predictor-Corrector Method in conjunction with a half-interval search technique which rapidly converges to the exact solution. Radial stress and displacement boundary conditions may be specified. Correlation of numerical integration results with analytical and finite-element solutions was found to be excellent. This analysis capability enables the determination of the influence of prescribed material property variations, temperature changes, and centrifugal forces on the response of an annular disk. The disk may be subjected to internal and external pressure or displacement boundary conditions as well. In addition, interference fit can be approximated by specifying displacement boundary conditions on the disk inner radius.

References

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2. R. L. McCullough, R. B. Pipes, D. Taggart and J. C. Mosko, "Influence of Fiber Orientation on the Properties of Short Fiber Composites," Composite Materials in the Automobile Industry, ASME, New York, 1978.
3. B. Carnahan et al, Applied Numerical Methods, Wiley and Sons, Inc., New York, 1969.
4. S. G. Lekhnitskii, Anisotropic Plates, translated by S. W. Tsai and T. Cheron, Gordon and Breach Science Publishers, New York, 1968.

Appendix A

Formulation of Governing Equations: Variable Properties

For an axisymmetric body rotating at a constant angular velocity ω , the equilibrium equation in the radial direction is given by:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0 \quad (1)$$

The stress - strain relations in polar coordinates for a cylindrically orthotropic material are given as follows:

$$\epsilon_r = \frac{\sigma_r}{E_r} - \frac{\nu_\theta \sigma_\theta}{E_\theta} + \alpha_r \Delta T \quad (2)$$

$$\epsilon_\theta = \frac{\sigma_\theta}{E_\theta} - \frac{\nu_r \sigma_r}{E_r} + \alpha_\theta \Delta T \quad (3)$$

Inverting (2) and (3) and solving in terms of the stress components yields,

$$\sigma_r = Q_{rr}(\epsilon_r - \alpha_r \Delta T) + Q_{r\theta}(\epsilon_\theta - \alpha_\theta \Delta T) \quad (4)$$

$$\sigma_\theta = Q_{r\theta}(\epsilon_r - \alpha_r \Delta T) + Q_{\theta\theta}(\epsilon_\theta - \alpha_\theta \Delta T) \quad (5)$$

where

$$Q_{rr} = \frac{E_r}{1 - \nu_{\theta r} \nu_{r\theta}} \quad Q_{\theta\theta} = \frac{E_\theta}{1 - \nu_{\theta r} \nu_{r\theta}} \quad (6)$$

$$Q_{r\theta} = Q_{\theta r} = \frac{E_r \nu_{\theta r}}{1 - \nu_{\theta r} \nu_{r\theta}} = \frac{E_\theta \nu_{r\theta}}{1 - \nu_{\theta r} \nu_{r\theta}} \quad (7)$$

The strain - displacement relations for an axisymmetric body are

$$\epsilon_r = \frac{du}{dr} \quad (8)$$

$$\epsilon_\theta = \frac{u}{r} \quad (9)$$

Where u is the radial displacement. To obtain the governing equation in terms of displacement equations (4), (5), (8) and (9) are substituted into the equilibrium equation (1). Simplifying one obtains,

$$\frac{d^2u}{dr^2} + S(r) \frac{du}{dr} + T(r)u = F(r) \quad (10)$$

where

$$S(r) = \left(\frac{1}{Q_{rr}} \frac{dQ_{rr}}{dr} + \frac{1}{r} \right) \quad (11)$$

$$T(r) = \left(\frac{1}{rQ_{rr}} \frac{dQ_{r\theta}}{dr} - \frac{k^2}{r^2} \right) \quad (12)$$

$$F(r) = \frac{-\rho\omega^2 r}{Q_{rr}} + \left\{ \frac{1}{Q_{rr}} \frac{dQ_{rr}}{dr} + \frac{1-\nu_{\theta r}}{r} \right\} \alpha_r \quad (13)$$

$$+ \left[\frac{1}{Q_{rr}} \frac{dQ_{r\theta}}{dr} + \frac{k^2(\nu_{r\theta}-1)}{r} \right] \alpha_\theta$$

$$+ \frac{d\alpha_r}{dr} + \nu_{\theta r} \frac{d\alpha_\theta}{dr} \} \Delta T$$

$$k^2 = E_\theta/E_r$$

and employing equations (6) and (7),

$$\frac{1}{Q_{rr}} \frac{dQ_{rr}}{dr} = \frac{1}{E_r} \frac{dE_r}{dr} + \frac{v_{r\theta}}{1-v_{\theta r}v_{r\theta}} \frac{dv_{\theta r}}{dr} + \frac{v_{\theta r}}{1-v_{\theta r}v_{r\theta}} \frac{dv_{r\theta}}{dr} \quad (14)$$

$$\frac{1}{Q_{r\theta}} \frac{dQ_{r\theta}}{dr} = \frac{v_{\theta r}}{E_r} \frac{dE_r}{dr} + \left(\frac{1}{1-v_{r\theta}v_{\theta r}} \right) \left[\frac{dv_{\theta r}}{dr} + v_{\theta r}^2 \frac{dv_{r\theta}}{dr} \right] \quad (15)$$

Reduction of equation (10) to a first order system follows:

$$\text{Let } u_1 = u$$

$$u_2 = \frac{du}{dr}$$

$$\text{then } \frac{du_1}{dr} = \frac{du}{dr} = u_2 \quad (16)$$

$$\frac{du_2}{dr} = \frac{d^2u}{dr^2}$$

Therefore (10) becomes

$$\frac{du_1}{dr} = u_2 \quad (17)$$

$$\frac{du_2}{dr} = F - S u_2 - T u_1 \quad (18)$$

and the initial conditions required at the inner radius "a" for Hamming's Method are,

$$\begin{aligned} u_1(a) &= u(a) \\ u_2(a) &= \frac{du}{dr}(a) \end{aligned} \quad (19)$$

For the original boundary value problem, one boundary condition (displacement or radial stress component) will be prescribed on the inner and outer radii. Consequently, the unknown initial condition will be bounded and the half-interval method will be employed to iterate to the solution. Note that Hamming's Method always requires boundary conditions at the inner radius in terms of displacement (see eq. (19)). Therefore, if $u(a)$ is prescribed, then bounds on $\frac{du}{dr}(a)$ are established and the solution is obtained in a straightforward manner. However, if $\sigma_r(a)$ is prescribed, bounds on $\sigma_\theta(a)$ are expected. In this case, the stress components are transformed into the displacement and radial strain boundary conditions utilizing equations (2), (3), (8) and (9).

The listing of the program which performs the numerical integration scheme is given below.

```

#FILE (1345)VARPROP/NUMD ON PACK
1000 $RESET FREE
1100 FILE 6(KIND=REMOTE,MAXRECSIZE=22)
1200 C
1300 C      HAMMING'S PREDICTOR-CORRECTOR METHOD
1400 C
1500 C      PROGRAM SOLVES A SYSTEM OF N FIRST ORDER ORDINARY DIFFERENTIAL
1600 C      EQUATIONS
1700 C
1800 C      LIST OF SYMBOLS:
1900 C
2000 C          XI      INITIAL STARTING VALUE FOR INDEPENDENT VARIABLE
2100 C          H        INTEGRATION STEP-SIZE
2200 C          XMAX     UPPER LIMIT OF INTEGRATION
2300 C          N        NUMBER OF INITIAL CONDITIONS
2400 C          YR(I)    N INITIAL CONDITIONS AT X
2500 C          FR(I)    DERIVATIVES IN ORIGINAL SYSTEM OF DIFFERENTIAL
2600 C                   EQUATIONS
2700 C          Y(I,J)    Y VALUE AT ITH X VALUE FOR JTH DIFFERENTIAL EQUATION
2800 C          F(I,J)    DERIVATIVE AT ITH X VALUE FOR JTH DIFFERENTIAL
2900 C                   EQUATION
3000 C          TE(I)    TRUNCATION ERROR FOR ITH CORRECTOR EQUATION
3100 C          INT       NUMBER OF INTEGRATIONS BETWEEN OUTPUT
3200 C          NN        UPPER LIMIT ON THE NUMBER OF HALF-INTERVAL ITERATIONS
3300 C          Y2LEFT    LOWER LIMIT ON UNKNOWN INITIAL VALUE
3400 C          Y2RITE    UPPER LIMIT ON UNKNOWN INITIAL VALUE
3500 C          NHIS      NUMBER OF HALF-INTERVAL ITERATIONS BETWEEN PRINTOUT
3600 C          SGI       RADIAL STRESS B.C. ON INNER RADIUS
3700 C          SGO       RADIAL STRESS B.C. ON OUTER RADIUS
3800 C          SIGNL     HAS THE VALUE -1, IF Y1(XMAX)<SIGO WHEN
3900 C                   Y2(XI)=Y2LEFT; OTHERWISE THE VALUE IS 1
4000 C          DT        TEMPERATURE DIFFERENCE RELATIVE TO STRESS
4100 C                   FREE STATE
4200 C          W          ROTATIONAL SPEED(RPM)
4300 C          RHO        DENSITY(#/IN**3)
4400 C
4500 C      SUBROUTINES RUNGE AND HAMMING ARE BASED ON ALGORITHM'S
4600 C      FOUND IN "APPLIED NUMERICAL METHODS" BY CARNAHAN, LUTHER
4700 C      AND WILKES (PAGES 367-402)
4800 C
4900 C      MAIN PROGRAM
5000 C
5100 C      INTEGER COUNT,RUNGE,HAMING
5200 C      LOGICAL PRED
5300 C      DIMENSION TE(10),YR(10),FR(10),Y(4,10),F(3,10),PHI(10),SAVEY(10)
5400 C      1,YPRED(10)
5500 C      COMMON DT,W,RHO,XI,XMAX
5600 C
5700 C      READ INPUT DATA

```

```

5800 C
5900 N=2
6000 WRITE(6,10)
6100 READ(5,/)XI,H,XMAX,INT,DT,W,RHO
6200 WRITE(6,20)
6300 READ(5,/)YR1,TIC1
6400 WRITE(6,25)
6500 READ(5,/)Y2LEFT,Y2RITE
6600 WRITE(6,26)
6700 READ(5,/)YR3,TIC3
6800 WRITE(6,27)
6900 READ(5,/)NN,NHIS,SIGNL
7000 C
7100 C PRINT HEADING AND INPUT DATA
7200 C
7300 WRITE(6,30)H,XI,XMAX,N,INT,YR1,TIC1,Y2LEFT,Y2RITE,NN,NHIS,SIGNL
7400 1 ,DT,RHO,W
7500 WRITE(6,40)
7600 W=W*2.*3.1415927/60.
7700 RHO=RHO/32.2/12.
7800 C
7900 C INITIALIZE STEP COUNTER
8000 C SET FIRST ROW OF Y MATRIX EQUAL TO INITIAL VALUES
8100 C INITIALIZE TRUNCATION ERRORS TO ZERO
8200 C
8300 DO 90 II=1,NN
8400 X=XI
8500 YR(1)=YR1
8600 Y2ZERO=(Y2LEFT+Y2RITE)/2.
8700 YR(2)=Y2ZERO
8800 WRITE(6,35)II,Y2LEFT,Y2RITE,Y2ZERO
8900 C
9000 C CONVERT STRESS B.C. TO DISPLACEMENT B.C.
9100 C
9200 IF(TIC1.EQ.2)GO TO 55
9300 CALL PROP(X,YR(1),YR(2),FF,P,Q,YR(2),EPT,YR(1),1)
9400 55 COUNT=0.
9500 M=0.
9600 DO 60 J=1,N
9700 TE(J)=0.
9800 60 Y(4,J)=YR(J)
9900 C
10000 C CALL FOURTH-ORDER RUNGE-KUTTA FUNCTION TO INTEGRATE
10100 C OVER FIRST THREE STEPS
10200 C SUBROUTINE DERIV CALCULATES DERIVATIVES
10300 C
10400 65 IF(RUNGE(M,N,YR,FR,X,H,PHI,SAVEY).NE.1)GOTO 70
10500 CALL DERIV(N,1,ISUB,YR,Y,FR,F,X)
10600 GOTO 65
10700 C
10800 C PUT APPROPRIATE INITIAL VALUES IN Y AND F. MATRICES

```

```

10900 70 COUNT=COUNT+1
11000 ISUB=4-COUNT
11100 DO 75 J=1,N
11200 75 Y(ISUB,J)=YR(J)
11300 CALL DERIV(N,2,ISUB,YR,Y,FR,F,X)
11400 C
11500 C PRINT SOLUTIONS AFTER INT STEPS
11600 C
11700 80 IF(COUNT/INT*INT.NE.COUNT)GO TO 85
11800 IF(.NOT.(II/NHIS*NHIS.EQ.II.OR.II.EQ.NN))GO TO 85
11900 IF(COUNT.LE.3)EPT=Y(ISUB,1)/X
12000 IF(COUNT.LE.3)CALL PROP(X,F(ISUB,1),EPT,FF,P,Q,SIGR,SIGT,U,3)
12100 IF(COUNT.GT.3)EPT=Y(1,1)/X
12200 IF(COUNT.GT.3)CALL PROP(X,F(1,1),EPT,FF,P,Q,SIGR,SIGT,U,3)
12300 IF(COUNT.LE.3)WRITE(6,50)X,Y(ISUB,1),F(ISUB,1),EPT,SIGR,SIGT
12400 IF(COUNT.GT.3)WRITE(6,50)X,Y(1,1),F(1,1),EPT,SIGR,SIGT
12500 C
12600 C IF X > XMAX TERMINATE INTEGRATION
12700 C
12800 85 CONTINUE
12900 IF(X.GT.XMAX-H/2)GOTO 100
13000 C
13100 C CALL RUNGE OR HAMMING TO INTEGRATE NEXT STEP
13200 C
13300 IF(COUNT.LT.3)GOTO 65
13400 PRED=.TRUE.
13500 105 MM=HAMING(N,Y,F,X,H,TE,PRED,YPRED)
13600 CALL DERIV(N,3,ISUB,YR,Y,FR,F,X)
13700 IF(MM.EQ.1)GOTO 105
13800 C
13900 C INCREMENT STEP COUNTER AND CONTINUE INTEGRATION
14000 COUNT=COUNT+1
14100 GO TO 80
14200 C
14300 C COMPARE SOLUTION TO KNOWN OUTER B.C.
14400 C
14500 100 IF(TIC3.EQ.2)GO TO 102
14600 C
14700 C CONVERT DISP. AND STRAIN TO STRESS
14800 C
14900 EPT=Y(1,1)/X
15000 EPR=F(1,1)
15100 CALL PROP(X,EPR,EPT,FF,P,Q,SIGR,SIGT,U,3)
15200 IF((SIGR-YR3)*SIGNL.GT.0)GO TO 106
15300 Y2RITE=Y2ZERO
15400 GO TO 90
15500 106 Y2LEFT=Y2ZERO
15600 GO TO 90
15700 102 IF((Y(1,1)-YR3)*SIGNL.GT.0)GO TO 110
15800 Y2RITE=Y2ZERO
15900 GO TO 90

```

```

16000 110 Y2LEFT=Y2ZERO
16100 90 CONTINUE
16200 C
16300 C   FORMAT STATEMENTS
16400 C
16500 10  FORMAT(1X,'ENTER XI,H,XMAX,INT,DT,W(RPM),RHO(LB/IN**3)')
16600 20  FORMAT(1X,'ENTER KNOWN I.C.,TYPE(STRESS=1,DISPL.=2)')
16700 25  FORMAT(1X,'ENTER Y2LEFT,Y2RITE')
16800 26  FORMAT(1X,'ENTER SECOND B.C.,TYPE(STRESS=1,DISPL.=2)')
16900 27  FORMAT(1X,'ENTER NN,NHIS,SIGNL')
17000 30  FORMAT(///1X,'H      =',E15.3/1X,'XI      =',F15.4/
17100 1  1X,'XMAX  =',F15.4/1X,'N      =',I15/1X,'INT    =',I15/
17200 1  1X,'YR1   =',F15.4/1X,'TIC1  =',F15.4/1X,'Y2LEFT=',F15.4/
17300 1  1X,'Y2RITE=',F15.4/1X,'NN    =',I15/1X,'NHIS   =',I15/
17400 1  1X,'SIGNL =',I15/1X,'DT     =',F15.4/1X,'RHO    =',F15.4/1X,
17500 1  'W      =',F15.4//)
17600 35  FORMAT(1X,'ITERATION',I5,5X,'LOWER=',F15.4,5X,'UPPER=',F15.4,
17700 1  5X,'UNKNOWN I.C.=',F15.4)
17800 40  FORMAT(7X,'X',30X,'Y',/30X,'U',18X,'EPR',16X,'EPT',16X
17900 1  ',SIGH',16X,'SIGT'//)
18000 50  FORMAT(1X,F10.5,5X,5(5X,E15.7))
18100 CALL EXIT
18200 END
18300 C*****
18400 FUNCTION RUNGE(M,N,Y,F,X,H,PHI,SAVEY)
18500 C
18600 INTEGER RUNGE
18700 DIMENSION PHI(N),SAVEY(N),Y(N),F(N)
18800 M=M+1
18900 GO TO (1,2,3,4,5), M
19000 C
19100 1  RUNGE=1
19200 RETURN
19300 C
19400 2  DO 22 J=1,N
19500 SAVEY(J)=Y(J)
19600 PHI(J)=F(J)
19700 22  Y(J)=SAVEY(J)+.5*H*F(J)
19800 X=X+.5*H
19900 RUNGE=1
20000 RETURN
20100 C
20200 3  DO 33 J=1,N
20300 PHI(J)=PHI(J)+2.*F(J)
20400 33  Y(J)=SAVEY(J)+.5*H*F(J)
20500 RUNGE=1
20600 RETURN
20700 C
20800 4  DO 44 J=1,N
20900 PHI(J)=PHI(J)+2.*F(J)
21000 44  Y(J)=SAVEY(J)+H*F(J)

```

```

21100      X=X+.5*H
21200      RUNGE=1
21300      RETURN
21400 C
21500      5 DO 55 J=1,N
21600      55 Y(J)=SAVEY(J)+(PHI(J)+F(J))*H/6.
21700      M=0.
21800      RUNGE=0
21900      RETURN
22000      END
22100 C*****
22200 C
22300      SUBROUTINE DERIV(N,FCOUNT,ISUB,YR,Y,FR,F,X)
22400 C
22500      DIMENSION YR(N),Y(4,N),FR(N),F(3,N)
22600      CALL PROP(X,SIGR,SIGT,FF,P,Q,EPR,EPT,U,2)
22700      GO TO(1,2,3), FCOUNT
22800      1 FR(1)=YR(2)
22900      FR(2)=FF-P*YR(2)-Q*YR(1)
23000      RETURN
23100      2 F(ISUB,1)=YR(2)
23200      F(ISUB,2)=FF-P*YR(2)-Q*YR(1)
23300      RETURN
23400      3 F(1,1)=Y(1,2)
23500      F(1,2)=FF-P*Y(1,2)-Q*Y(1,1)
23600      RETURN
23700      END
23800 C*****
23900 C
24000      FUNCTION HAMING(N,Y,F,X,H,TE,PRED,YPRED)
24100 C
24200      INTEGER HAMING
24300      LOGICAL PRED
24400      DIMENSION YPRED(N),TE(N),Y(4,N),F(3,N)
24500 C      IS CALL FOR PREDICTION OR CORRECTOR SECTION
24600      IF(.NOT.PRED) GOTO 4
24700 C
24800 C      PREDICTOR SECTION OF HAMING
24900 C      COMPUTE PREDICTED VALUES AT NEXT POINT
25000      DO 1 J=1,N
25100      1 YPRED(J)=Y(4,J)+4.*H*(2.*F(1,J)-F(2,J)+2.*F(3,J))/3.
25200 C
25300 C      UPDATE THE Y AND F TABLES
25400      DO 2 J=1,N
25500      DO 2 K5=1,3
25600      K=5-K5
25700      Y(K,J)=Y(K-1,J)
25800      IF(K.LT.4)F(K,J)=F(K-1,J)
25900      2 CONTINUE
26000 C
26100 C      MODIFY PREDICTED Y(J) VALUES USING THE TRUNCATION ERROR

```

```

26200 C      ESTIMATES FROM THE PREVIOUS STEP, INCREMENT X VALUE
26300      DO 3 J=1,N
26400 3      Y(1,J)=YPRED(J)+112.*TE(J)/9.
26500      X=X+H
26600 C
26700 C      SET PRED AND REQUEST UPDATED DERIVATIVE VALUES
26800      PRED=.FALSE.
26900      HAMING=1
27000      RETURN
27100 C
27200 C      CORRECTOR SECTION OF HAMING
27300 C      COMPUTE CORRECTED AND IMPROVED VALUES OF THE Y(J) AND
27400 C      SAVE TRUNCATION ERROR ESTIMATES FOR THE CURRENT STEP
27500 4      DO 5 J=1,N
27600      Y(1,J)=(9.*Y(2,J)-Y(4,J)+3.*H*(F(1,J)+2.*F(2,J)-F(3,J)))/8.
27700      TE(J)=9.*(Y(1,J)-YPRED(J))/121.
27800 5      Y(1,J)=Y(1,J)-TE(J)
27900 C
28000 C      SET PRED AND RETURN WITH SOLUTIONS FOR CURRENT STEP
28100      PRED=.TRUE.
28200      HAMING=2
28300      RETURN
28400      END
28500 C*****
28600 C
28700      SUBROUTINE PROP(X,A1,A2,FF,F,Q,A3,A4,A5,OPT1)
28800 C
28900 C      INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION X
29000 C      (T: TANGENTIAL, R: RADIAL):
29100 C
29200 C      ET  -TANGENTIAL MODULI
29300 C      ETP -DERIVATIVE OF ET
29400 C      VTR -POISSON RATIO
29500 C      VTRP -DERIVATIVE OF VTR
29600 C      ER  -RADIAL MODULI
29700 C      ERP -DERIVATIVE OF ER
29800 C      VTR -POISSON RATIO
29900 C      VTRP -DERIVATIVE OF VTR
30000 C      AT  -TANGENTIAL COEFFICIENT OF THERMAL EXPANSION
30100 C      ATP -DERIVATIVE OF AT
30200 C      AR  -RADIAL COEFFICIENT OF THERMAL EXPANSION
30300 C      DT  -TEMPERATURE CHANGE(POSITIVE VALUE CORRESPONDS TO AN INCREASE
30400 C           RELATIVE TO THE STRESS FREE STATE)
30500 C      EPR -RADIAL STRAIN COMPONENT
30600 C      EPT -TANGENTIAL STRAIN COMPONENT
30700 C      U    -RADIAL DISPLACEMENT
30800 C
30900      REAL K
31000      COMMON DT,W,RHO,XI,XMAX
31100 C
31200 C      INPUT MATERIAL PROPERTY DEPENDENCE ON RADIAL POSITION

```

```

31300 C
31400     ETM=1.4E6
31500     VTR=.27
31700     ERM=2.6E6
31900     V RTP=0.
32000     ATM=10.1E-6
32100     ARM=4.5E-6
32200     ET=(ERM-ETM)*X/(XMAX-XI)+ETM-(ERM-ETM)*XI/(XMAX-XI)
32300     ER=(ETM-ERM)*X/(XMAX-XI)+ERM-(ETM-ERM)*XI/(XMAX-XI)
32400     AR=(ATM-ARM)*X/(XMAX-XI)+ARM-(ATM-ARM)*XI/(XMAX-XI)
32500     AT=(ARM-ATM)*X/(XMAX-XI)+ATM-(ARM-ATM)*XI/(XMAX-XI)
32600     ERP=(ETM-ERM)/(XMAX-XI)
32700     ETP=(ERM-ETM)/(XMAX-XI)
32800     ATP=(ARM-ATM)/(XMAX-XI)
32900     ARP=(ATM-ARM)/(XMAX-XI)
32950     VRT=ER*VTR/ET
32960     V RTP=VTR*ERP/ET-VTR*ER/ET**2*ETP
33000 C  END MATERIAL PROPERTY INPUT
33100     VD=1.-VRT*VTR
33200     QRR=ER/VD
33300     QTT=ET/VD
33400     QRT=ER*VTR/VD
33500     QRRP=ERP/ER+VRT*VTRP/VD+VTR*V RTP/VD
33600     QRTP=VTR*ERP/ER+(VTRP+VTR**2*V RTP)/VD
33700     K=SQRT(ET/ER)
33800     IF(OPT1,EQ,1)GO TO 1
33900     IF(OPT1,EQ,3)GO TO 3
34000     FF=-RHO*W**2*X/QRR+((QRRP+(1-VTR)/X)*AR
34100 1   +(QRTP+K**2*(VRT-1.)/X)*AT+ARP+VTR*ATP)*DT
34200     P=1./X+QRRP
34300     Q=QRTP/X-(K/X)**2
34400     IF(OPT1,EQ,2)GO TO 2
34500 1   A3=A1/ER-VTR*A2/ET+AR*DT
34600     A4=-VRT*A1/ER+A2/ET+AT*DT
34700     A5=X*A4
34800     GO TO 2
34900 3   A3=QRR*A1+QRT*A2-DT*(QRR*AR+QRT*AT)
35000     A4=QRT*A1+QTT*A2-DT*(QRT*AR+QTT*AT)
35100 2   RETURN
35200     END
#

```


Appendix B

Analytic Solution: Constant Properties

For the case of constant properties, equation (10) can be rewritten as follows:

$$r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} - k^2 u = C_1 r - C_2 r^3 \quad (20)$$

$$\text{where } C_1 = \Delta T \{ 2_r(1-\nu_{\theta r}) + k^2 \alpha_{\theta}(\nu_{r\theta}-1) \}$$

(21)

$$C_2 = \frac{(1-\nu_{\theta r}\nu_{r\theta})}{E_r} \rho \omega^2$$

Noting that (20) is in the form of Euler's equations, the general solution can be expressed as follows:

$$u = A r^k + B r^{-k} + \frac{C_1 r}{1-k^2} - \frac{C_2 r^3}{9-k^2} \quad (k \neq 1 \text{ or } 3) \quad (22)$$

where A and B are constants to be determined from the boundary conditions. The corresponding stress components are:

$$\begin{aligned} \sigma_r = & \frac{A E_r(k+\nu_{\theta r})}{1-\nu_{\theta r}\nu_{r\theta}} r^{k-1} + \frac{B E_r(\nu_{\theta r}-k)}{1-\nu_{\theta r}\nu_{r\theta}} r^{-k-1} \\ & - \frac{(3+\nu_{\theta r})\rho \omega^2 r^2}{9-k^2} + \frac{\Delta T E_{\theta}}{(1-k^2)} (\alpha_r - \alpha_{\theta}) \end{aligned} \quad (23)$$

$$\sigma_{\theta} = \frac{A E_{\theta} (1 + k \nu_{r\theta})}{(1 - \nu_{r\theta} \nu_{\theta r})} r^{k-1} + \frac{B E_{\theta} (1 - k \nu_{r\theta})}{(1 - \nu_{r\theta} \nu_{\theta r})} r^{-k-1} - \frac{(1 + 3 \nu_{r\theta}) k^2 \rho \omega^2 r^2}{9 - k^2} + \frac{\Delta T E_{\theta}}{1 - k^2} (\alpha_r - \alpha_{\theta}) \quad (24)$$

If radial stress boundary conditions are prescribed on the inner (a) and outer radius (b),

$$\sigma_r(a) = p \quad (25)$$

$$\sigma_r(b) = q$$

The unknown constants are,

$$A = \frac{(P d^{k+1} - Q)}{b^{k-1} [d^{2k} - 1] Q_{rr} (k + \nu_{\theta r})} \quad (26)$$

$$B = \frac{(-P + d^{k-1} Q) d^{k+1}}{b^{-k-1} [d^{2k} - 1] Q_{rr} (\nu_{\theta r} - k)}$$

$$\text{where } P = p + \frac{(3 + \nu_{\theta r}) \rho \omega^2 a^2}{9 - k^2} - \frac{\Delta T E_{\theta} (\alpha_r - \alpha_{\theta})}{1 - k^2} \quad (27)$$

$$Q = q + \frac{(3 + \nu_{\theta r}) \rho \omega^2 b^2}{9 - k^2} - \frac{\Delta T E_{\theta} (\alpha_r - \alpha_{\theta})}{1 - k^2}$$

$$d = a/b, \quad k^2 = E_{\theta}/E_r$$

Employing equation (26), we obtain the following relations for the stress components:

$$\sigma_r = \frac{Pd^{k+1}-Q}{d^{2k-1}} \left(\frac{r}{b}\right)^{k-1} - \frac{(P-d^{k-1}Q)d^{k+1}}{d^{2k-1}} \left(\frac{r}{b}\right)^{-k-1} \quad (28)$$

$$- \frac{(3+v_{\theta r})\rho\omega^2 r^2}{9-k^2} + \frac{\Delta TE_{\theta}}{(1-k^2)} (\alpha_r - \alpha_{\theta})$$

$$\sigma_{\theta} = \frac{k^2(1+k v_{r\theta})}{(k+v_{\theta r})} \frac{(Pd^{k+1}-Q)}{(d^{2k-1})} \left(\frac{r}{b}\right)^{k-1}$$

$$- \frac{k^2(1-k v_{r\theta})}{(v_{\theta r}-k)} \frac{(P-d^{k-1}Q)}{(d^{2k-1})} d^{k+1} \left(\frac{r}{b}\right)^{-k-1} \quad (29)$$

$$- \frac{(1+3 v_{r\theta})k^2\rho\omega^2 r^2}{9-k^2} + \frac{\Delta TE_{\theta}}{(1-k^2)} (\alpha_r - \alpha_{\theta})$$

A program listing for the analytical solution derived for radial stress boundary conditions is given below. Stress components are determined directly from equations (28) and (29) and strain components and radial displacement are calculated from equations (2), (3) and (9), respectively.

#FILE (1345)VARPROP/CFD ON PACK

1000 \$RESET FREE

1100 FILE 6(KIND=REMOTE,MAXRECSIZE=22)

1200 REAL K

1300 C LIST OF SYMBOLS:

1400 C
1500 C XI INITIAL STARTING VALUE FOR INDEPENDENT VARIABLE
1600 C DX X INCREMENT
1700 C XMAX UPPER LIMIT OF INTEGRATION
1800 C SGI RADIAL STRESS B.C. ON INNER RADIUS
1900 C SGO RADIAL STRESS B.C. ON OUTER RADIUS
2000 C
2100 C ET -TANGENTIAL MODULI
2300 C VTR -POISSON RATIO
2500 C ER -RADIAL MODULI
2600 C AT -TANGENTIAL COEFFICIENT OF THERMAL EXPANSION
2800 C AR -RADIAL COEFFICIENT OF THERMAL EXPANSION
2900 C DT -TEMPERATURE CHANGE(POSITIVE VALUE CORRESPONDS TO AN INCREASE
3000 C RELATIVE TO THE STRESS FREE STATE)
3100 C EPR -RADIAL STRAIN COMPONENT
3200 C EPT -TANGENTIAL STRAIN COMPONENT
3300 C U -RADIAL DISPLACEMENT
3350 C W -ROTATIONAL SPEED(RPM)
3360 C RHO -DENSITY(#/IN**3)

3400 C
3500 C INPUT MATERIAL PROPERTIES

3600 C (T: TANGENTIAL, R:RADIAL):

3700 C
3800 ET=2.6E6
4000 VTR=.501
4200 ER=1.4E6
4250 VRT=VTR*ER/ET
4300 AT=4.5E-6
4500 AR=10.1E-6
4550 RHO=.1

4600 C
4700 C END MATERIAL PROPERTY DESCRIPTION

4800 C
4900 C READ INPUT DATA

5000 C
5100 WRITE(6,10)
5200 READ(5,/)XI,XMAX,DX,DT,W
5300 WRITE(6,20)
5400 READ(5,/)SGI,SGO

5500 C
5600 C PRINT HEADING AND INPUT DATA

5700 C
5800 WRITE(6,30)XI,XMAX,SGO,SGI,ET,ER,AT,AR,VTR,DT,W,RHO
5900 WRITE(6,40)

```

6000 C
6010 W=W*2*3.1415927/60.
6020 RHO=RHO/32.2/12.
6100 D=XI/XMAX
6200 K=SQRT(ET/ER)
6400 DEN=D**2/K-1.
6402 P1=DT*(AR-AT)/(1.-K**2)
6404 P2=(3+VTR)*RHO*W**2/(9.-K**2)
6406 P=SGI+P2*X**2-P1*K**2*ER
6408 Q=SGO+P2*XMAX**2-P1*K**2*ER
6500 X=XI
6505 80 R=X/XMAX
6510 SIGR=(P*D**2*(K+1)-Q)*R**2*(K-1)/DEN-(P-D**2*(K-1)*Q)*D**2*(K+1)
6515 1 /DEN*R**2*(-K-1)-P2*X**2+P1*K**2*ER
6520 SIGT=K**2*(1+K*VTR)/(K+VTR)*(P*D**2*(K+1)-Q)/DEN*R**2*(K-1)
6525 1 -K**2*(1-K*VTR)/(VTR-K)*(P-D**2*(K-1)*Q)/DEN*D**2*(K+1)
6530 1 *R**2*(-K-1)-(1+3.*VTR)*RHO*(W*K*X)**2/(9-K**2)+P1*ET
8000 70 EPR=SIGR/ER-VTR*SIGT/ET+AR*DT
8100 EPT=-VTR*SIGR/ET+SIGT/ET+AT*DT
8200 U=X*EPT
8300 WRITE(6,50)X,SIGR,SIGT,EPR,EPT,U
8400 IF(X.GE.XMAX-DX)GO TO 2
8500 X=X+DX
8600 GO TO 80
8700 C
8800 C FORMAT STATEMENTS
8900 C
9000 10 FORMAT(1X,'ENTER XI,XMAX,DX,DT,W')
9100 20 FORMAT(1X,'ENTER SGI,SGO')
9200 30 FORMAT(///1X,'XI =',E15.3/1X,'XMAX =',E15.4/
9300 1 1X,'SGO =',E15.4/1X,'SGI =',E15.4/1X,'ET =',E15.4/
9400 1 1X,'ER =',E15.4/1X,'AT =',E15.4/1X,'AR =',E15.4/
9500 1 1X,'VTR =',E15.3/1X,'DT =',E15.4/1X,'W =',E15.4/
9505 1 1X,'RHO =',E15.4//)
9600 40 FORMAT(17X,'X',14X,'SIGR',16X,'SIGT',16X,'EPR',17X
9700 1 ', 'EPT',18X,'U'//)
9800 50 FORMAT(10X,F10.4,5(5X,E15.7))
9900 2 RETURN
10000 END
#

```

Appendix C

Analytical Solution: Power Law Variation of Properties

For the power law variation of moduli, the stress function approach [4] is utilized to obtain a closed form solution. Employing the strain-displacement relations in (8) and (9) and eliminating u in (2) and (3), we obtain

$$\begin{aligned} & \left(-\frac{rv_{\theta r}}{E_r} \sigma_r' \right) + \left(\frac{r}{E_{\theta}} \sigma_{\theta}' \right) + (r\alpha_{\theta}\Delta T) \\ &= -\frac{\sigma_r}{E_r} - \frac{v_{\theta r}}{E_{\theta}} \sigma_{\theta}' + \alpha_r \Delta T \end{aligned} \quad (30)$$

where prime denotes derivatives with respect to radial position.

For an axisymmetric body, the relations between the stress function and stress components which identically satisfy Eq. (1), (in the absence of body forces) are

$$\sigma_r = g/r, \quad \sigma_{\theta} = g' \quad (31)$$

Substitution of (26) into (25) yields the governing differential equation,

$$\begin{aligned} g'' + \left(\frac{1}{r} - \frac{E_{\theta}'}{E_{\theta}} \right) g' + \left(\frac{v_{\theta r} E_{\theta}'}{r E_{\theta}} - \frac{v_{\theta r}'}{r} - \frac{E_{\theta}}{r^2 E_r} \right) g \\ = -\Delta T \left[\frac{E_{\theta} (\alpha_{\theta} - \alpha_r)}{r} + E_{\theta} \alpha_{\theta}' \right] \end{aligned} \quad (32)$$

Assuming the exponential form for the moduli equation (32) may be ordered

$$E_r = E_{rm} r^m, E_\theta = E_{\theta m} r^m \quad (33)$$

$$v_{\theta r} = \text{constant}, v_{r\theta} = v_{\theta r} \frac{E_{rm}}{E_{\theta m}}$$

$$r^2 g'' + r(1-m)g' + (mv_{\theta r} - k^2)g$$

$$= - \Delta T E_{\theta m} r^{m+2} \left[\frac{(\alpha_\theta - \alpha_r)}{r} + \alpha_\theta' \right] \quad (34)$$

The general solution of equation (29) can be expressed as,

$$g = Ar^{N_1} + Br^{N_2} + g_p \quad (35)$$

where $N_{1,2} = \frac{m \pm \sqrt{m^2 + 4(k^2 - mv_{\theta r})}}{2}$

and g_p is the particular solution.

The particular solution is determined by a variation of parameter's approach,

$$g_p = v_1 r^{N_1} + v_2 r^{N_2} \quad (36)$$

where

$$v_1' = \frac{-r^{1-N_1} C_3}{(N_2 - N_1)} \quad (37)$$

$$v_2' = \frac{r^{1-N_2} C_3}{(N_2 - N_1)}$$

and C_3 is defined as the right-hand side of equation (34)

$$C_3 = -\Delta T E_{\theta m} r^{m+2} \left[\frac{(\alpha_{\theta} - \alpha_r)}{r} + \alpha_{\theta}' \right] \quad (38)$$

The stress components defined in equation (31) are:

$$\sigma_r = A r^{N_1-1} + B r^{N_2-1} + r^{N_1-1} v_1 + r^{N_2-1} v_2 \quad (39)$$

$$\sigma_{\theta} = A N_1 r^{N_1-1} + B N_2 r^{N_2-1} + N_1 r^{N_1-1} v_1 + N_2 r^{N_2-1} v_2 \quad (40)$$

and A and B are unknown constants to be determined from the boundary conditions. The components of displacement and strain are calculated from equations (9), (2) and (3), respectively.

If radial stress boundary conditions are prescribed in equation (25), the unknown constants are given as follows:

$$A = \frac{P d^{1-N_2} - Q}{b^{N_1-1} (d^{N_1-N_2} - 1)} \quad (41)$$

$$B = \frac{-(P - d^{N_1-1} Q) d^{1-N_2}}{b^{N_2-1} (d^{N_1-N_2} - 1)} \quad (42)$$

where

$$P = p - a^{N_1-1} v_1(a) - a^{N_2-1} v_2(a) \quad (43)$$

$$Q = q - b^{N_1-1} V_1(b) - b^{N_2-1} V_2(b) \quad (44)$$

and the V_i ($i=1,2$) are obtained through integration of equation (37) and evaluation at the boundaries.

For completeness, assume the coefficients of thermal expansion also vary as a power law, i.e.,

$$\alpha_\theta = \alpha_{\theta m} r^m \quad (45)$$

$$\alpha_r = \alpha_{rm} r^m \quad (46)$$

Equation (38) reduces to

$$C_3 = C_4 r^{2m-1} \quad (47)$$

where

$$C_4 = -\Delta T E_{\theta m} [\alpha_{\theta m} (m+1) - \alpha_{rm}] \quad (48)$$

and integrating equation (37) yields

$$V_1 = \frac{-C_4 r^{2m-N_1+1}}{(N_2-N_1)(2m-N_1+1)} \quad (49)$$

$$V_2 = \frac{C_4 r^{2m-N_2+1}}{(N_2-N_1)(2m-N_2+1)} \quad (50)$$

Employing equations (41), (42), (43) and (44) we obtain the following expression for the stress components:

$$\sigma_r = \left[\frac{Pd^{1-N_2} - Q}{N_1 - N_2 - 1} \right] \left(\frac{r}{b} \right)^{N_1 - 1} - \left[\frac{P - d^{N_1 - 1} Q}{N_1 - N_2 - 1} \right] d^{1 - N_2} \left(\frac{r}{b} \right)^{N_2 - 1} + \frac{C_4 r^2}{(2m - N_1 + 1)(2m - N_2 + 1)} \quad (51)$$

$$\sigma_\theta = \left[\frac{Pd^{1-N_2} - Q}{N_1 - N_2 - 1} \right] N_1 \left(\frac{r}{b} \right)^{N_1 - 1} - \left[\frac{P - d^{N_1 - 1} Q}{N_1 - N_2 - 1} \right] N_2 d^{1 - N_2} \left(\frac{r}{b} \right)^{N_2 - 1} + \frac{C_4 r^{2m} (1 + 2m)}{(2m - N_1 + 1)(2m - N_2 + 1)} \quad (52)$$

Equations (43) and (44) simplify to the following:

$$P = p - \frac{C_4 a^{2m}}{(2m - N_1 + 1)(2m - N_2 + 1)} \quad (53)$$

$$Q = q - \frac{C_4 b^{2m}}{(2m - N_1 + 1)(2m - N_2 + 1)} \quad (54)$$

Note that for $m = 0$ ($N_1 = k$, $N_2 = -k$), equations (51) - (54) reduce to the solution given in Appendix B for constant material properties. The program listing for this solution is given next.

```

#FILE (1345)VARPROP/CF2 ON PACK
1000 $RESET FREE
1100 FILE 6(KIND=REMOTE,MAXRECSIZE=22)
1200 REAL K,N1,N2,M
1300 C LIST OF SYMBOLS:
1400 C
1500 C XI INITIAL STARTING VALUE FOR INDEPENDENT VARIABLE
1600 C DX X INCREMENT
1700 C XMAX UPPER LIMIT OF INTEGRATION
1800 C SGI RADIAL STRESS B.C. ON INNER RADIUS
1900 C SGO RADIAL STRESS B.C. ON OUTER RADIUS
2000 C
2100 C ETM -TANGENTIAL MODULI
2200 C VTR -POISSON RATIO
2300 C ERM -RADIAL MODULI
2400 C ATM -TANGENTIAL COEFFICIENT OF THERMAL EXPANSION
2500 C ARM -RADIAL COEFFICIENT OF THERMAL EXPANSION
2600 C DT -TEMPERATURE CHANGE(POSITIVE VALUE CORRESPONDS TO AN INCREASE
2700 C RELATIVE TO THE STRESS FREE STATE)
2800 C EPR -RADIAL STRAIN COMPONENT
2900 C EPT -TANGENTIAL STRAIN COMPONENT
3000 C U -RADIAL DISPLACEMENT
3100 C
3200 C INPUT MATERIAL PROPERTIES
3300 C (T: TANGENTIAL, R:RADIAL):
3400 C
3500 ETM=1.4E6
3600 VTR=.27
3700 ERM=2.6E6
3800 ATM=10.1E-6
3900 ARM=4.5E-6
4000 C
4100 C END MATERIAL PROPERTY DESCRIPTION
4200 C
4300 C READ INPUT DATA
4400 C
4500 WRITE(6,10)
4600 READ(5,/)XI,XMAX,DX,DT
4700 WRITE(6,20)
4800 READ(5,/)SGI,SGO,M
4900 C
5000 C PRINT HEADING AND INPUT DATA
5100 C
5200 WRITE(6,30)XI,XMAX,SGO,SGI,ETM,ERM,ATM,ARM,VTR,DT,M
5300 WRITE(6,40)

```